ON THE ANGLE BETWEEN TWO LINES GIVEN BY THEIR EQUATIONS.

BY W. E. HEAL, WHEELING, INDIANA.

WORKS on analytic geometry give the following expressions for the angle, φ, between two right lines whose equations are

$$A\ x+B\ y+C\ =0,$$
$$A_1x+B_1y+C_1=0.$$

$$\sin\varphi=\frac{(A_1B—AB_1)\sin\omega}{\sqrt{(A^2+B^2—2AB\cos\omega)}\sqrt{(A_1^2+B_1^2—2A_1B_1\cos\omega)}},$$

$$\cos\varphi=\frac{AA_1+BB_1—(AB_1+A_1B)\cos\omega}{\sqrt{(A^2+B^2—2AB\cos\omega)}\sqrt{(A_1^2+B_1^2-2A_1B_1\cos\omega)}};$$

$$\therefore\qquad \tan\varphi=\frac{(A_1B—AB_1)\sin\omega}{AA_1+BB_1—(AB_1+A_1B)\cos\omega}.$$

Therefore, say these authors, the lines are parallel if $(A_1B—AB_1)\sin\omega=0$, since the tagent then becomes zero. Likewise the lines are perpendicular if $AA_1+BB_1—(AB_1—A_1B)\cos\omega=0$, because the tangent then becomes infinite.

This reasoning is not conclusive, since it is not proved that these expressions may not both be zero for the same values of A, A_1, B, B_1, and thus the tangent be *indeterminate.*

I think a better proof of the above proposition might be given as follows.

The equation

$$[(A_1B—AB_1)\sin\omega]^2+[AA_1+BB_1—(AB_1+A_1B)\cos\omega]^2$$
$$=(A^2+B^2—2AB\cos\omega)(A_1^2+B_1^2—2A_1B_1\cos\omega)\quad(1)$$

is identical.

If the lines are parallel, $\cos\varphi=\cos 0°=1.\ \therefore$

$$AA_1+BB_1—(AB_1+A_1B)\cos\omega=\sqrt{(A^2—B^2—2AB\cos\omega)}$$
$$\times\sqrt{(A_1^2+B_1^2—2A_1B_1\cos\omega)}.\quad(2)$$

Squaring (2), subtracting from (1), extracting the square root and dividing by $\sin\omega$, we obtain for the condition of parallelism

$$A_1B—AB_1=0.$$

If the lines are perpendicular, $\sin\varphi=\sin 90°=1;\ \therefore$

$$(A_1B—AB_1)\sin\omega=\sqrt{(A^2+B^2—2AB\cos\omega)}\sqrt{(A_1^2+B_1^2—2A_1B_1\cos\omega)}.\quad\ldots(3)$$

Squaring (3), subtracting from (1) and extracting the square root we get for the condition of perpendicularity

$$AA_1+BB_1—(AB_1+A_1B)\cos\omega=0.$$